ASYMPTOTIC BEHAVIOUR OF A SEMILINEAR ELLIPTIC SYSTEM WITH A LARGE EXPONENT

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ABSTRACT. Consider the problem

$$\begin{aligned} -\Delta u &= v^{\frac{2}{N-2}}, \quad v>0 \quad \text{in} \quad \Omega, \\ -\Delta v &= u^p, \qquad u>0 \quad \text{in} \quad \Omega, \\ u &= v \ = \ 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N>2$, with smooth boundary $\partial\Omega$. We study the asymptotic behaviour of the least energy solutions of this system as $p\to\infty$. We show that the solution remain bounded for p large and have one or two peaks away form the boundary. When one peak occurs we characterize its location.

1. INTRODUCTION

In this article we consider the problem

$$\begin{cases} -\Delta(-\Delta u)^{(N-2)/2} = u^p, \quad u>0 \quad \text{in} \quad \Omega, \\ \qquad\qquad u = \Delta u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N>2$, with smooth boundary $\partial\Omega$. We consider the so-called least energy solutions of (1.1), obtained by the minimization problem

$$c_p := \inf_{v\in W^{2,\frac{N}{2}}(\Omega)\cap W_0^{2,\frac{N}{2}}(\Omega)} \left\{ \left(\int_\Omega |\Delta v|^{N/2}dx\right)^{2/N} \colon \|v\|_{p+1}=1 \right\}$$

By standard argument c_p is achieved by a positive function $\underline{u}_p$, which is a positive scalar multiple of a function solving (1.1). Let us denote such least energy solution by u_p.

Problem (1.1) is the particular case $q=2/(N-2)$ for the system

$$\begin{aligned} -\Delta u &= v^q, \quad v>0 \quad \text{in} \quad \Omega, \\ -\Delta v &= u^p, \quad u>0 \quad \text{in} \quad \Omega, \\ u &= v \ = \ 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

For this system the condition on (p,q), given by

$$\frac{N}{p+1}+\frac{N}{q+1}-(N-2)=0, \tag{1.2}$$

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so called *critical hyperbola*, describes a the borderline between existence and non-existence of positive solutions. In this article, we fix $q = 2/(N-2)$, that is we *stand* in an asymptote, and we prove that as we increase p the least energy solution develops a peak behavior. The case $N = 4$ these type of results were shown in [3, 14, 15] and in the case $N = 2$, the problem reduces to one equation, and we observe a similar behaviour, see [12, 13].

More precisely, our aim is to prove the following results.

Theorem 1.1. *Let u_p the least energy solution of* (1.1) *There exists C_1, C_2 independent of p such that*

$$0 < C_1 < \|u_p\|_{L^\infty(\Omega)} < C_2 < +\infty$$

for p large enough.

For the next result we define

$$w_p := \frac{u_p}{(\int\limits_\Omega u_p^p\, dx)^{\frac{2}{N-2}}}$$

For a sequence w_{p_n} of w_p, we define the *blow up set* S of $\{w_{p_n},\}$ as

$$\begin{aligned} S :=\{ & x \in \overline{\Omega} \colon \exists \text{ a subsequence } \quad w_{p_n}, \\ & \exists \{x_n\} \subset \Omega \text{ such that } x_n \to x \text{ and } w_{p_n}(x_n) \to \infty \}. \end{aligned}$$

We define a *peak point* P for u_p to be a point in $\overline{\Omega}$ such that u_p does not vanish in the L^∞ norm in any neighborhood of P as $p \to \infty$. We shall see later that peaks point of $\{u_p\}$ are contained in the blow up set S of $\{w_p\}$

Theorem 1.2. *Let Ω a convex bounded domain in $\mathbb{R}^N$, $N > 3$, with smooth boundary $\partial\Omega$. Then for any sequence w_{p_n} of w_p, with $p_n \to \infty$ there exists a sequence still denoted by w_{p_n} such that the blow up set S of this subsequence is contained in Ω and has the property $1 \le card(S) \le 2$.*

If $card(S) = 1$ and $S = \{x_0\}$ then:
1)

$$f_n := \frac{u_{p_n}^{p_n}}{\int\limits_\Omega u_{p_n}^{p_n}\, dx} = (\int\limits_\Omega u_{p_n}^{p_n}\, dx)^{\frac{2}{N-2}p_n - 1} w_{p_n}^{p_n} \to \delta_{x_0}.$$

in the sense of distributions.

2) $w_{p_n} \to \tilde{G}(\cdot, x_0)$ in $C^2_{loc}(\overline{\Omega} \setminus \{x_0\})$ where $\tilde{G}(x,y)$ solves

$$-\Delta \tilde{G}(x, \cdot) = G^{\frac{2}{N-2}}(x, \cdot) \quad \text{in} \quad \Omega, \qquad \tilde{G}(x, \cdot) = 0 \quad \text{on} \quad \partial\Omega,$$

where $-\Delta G(\cdot, y) = \delta_y$ in Ω, $G(\cdot, y) = 0$ on $\partial\Omega$.

3) x_0 is a critical point of $\tilde{\phi}(x) := \tilde{g}(x,x)$ where the function $\tilde{g}(x,y)$ is given by

$$\tilde{g}(x,y) = \tilde{G}(x,y) + \frac{1}{(N-2)^{\frac{N}{N-2}} \omega_{N-1}^{\frac{2}{N-2}}} \log|x-y|,$$

where ω_{N-1} the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

We observe that regularity of $\tilde{\phi}$ is needed to compute its critical points in 3). Indeed, by definition of $\tilde{G}$, we have

$$\lim_{y\to x}|x-y|^{4-N}\Delta\tilde{g}(x,y) = -\frac{2}{N-2}\frac{g(x,x)}{((N-2)\omega_{N-1})^{\frac{4-N}{N-2}}} \tag{1.3}$$

for $x\in\Omega$, where $g(x,y)$ is the regular part of $G(x,y)$, i.e

$$g(x,y) = G(x,y) - \frac{1}{(N-2)\omega_{N-1}|x-y|^{N-2}}.$$

By elliptic regularity, for $N\geq 3$, the function $g(x,\cdot)$ is regular and so is $\tilde{\phi}$.

Remark 1.3. *We conjecture that for $N=3$ the conclusions in Theorem 1.2 also hold. The only difficulty is to prove Lemma 3.3 for $N=3$, but we think that is only technical.*

2. Estimates for c_p

Lemma 2.1. *For every $t\geq N/2$. there is D_t such that*

$$\|u\|_t \leq D_t t^{\frac{N-2}{N}}\|\Delta u\|_{N/2}$$

where

$$\lim_{t\to\infty} D_t = (\frac{N-2}{Neb_0})^{\frac{N-2}{N}}$$

with $b_0 = \frac{N}{\omega_{N-1}}[4\pi^{N/2}/\Gamma((N-2)/2)]^{\frac{N}{N-2}} = N(N-2)^{\frac{N}{N-2}}\omega_{N-1}^{\frac{2}{N-2}}$.

Proof. From [1], we have the following Higher-Order version of the Moser-Trudinger inequality,

$$\int_\Omega \exp(b_0|u|^{N/(N-2)})dx \leq C|\Omega|$$

for any u such that $\|\Delta u\|_{N/2}\leq 1$. Therefore

$$\frac{1}{\Gamma(\frac{t(N-2)}{N}+1)}\int_\Omega u^t\,dx = \frac{1}{\Gamma(\frac{t(N-2)}{N}+1)}\int[b_0(\frac{u}{\|\Delta u\|_{N/2}})^{\frac{N}{N-2}}]^{\frac{t(N-2)}{N}}\,dx \tag{2.1}$$

$$\times\, b_0^{-\frac{t(N-2)}{N}}\|\Delta u\|_{N/2}^t \tag{2.2}$$

$$\leq \int_\Omega \exp(b_0(\frac{u}{\|\Delta u\|_{N/2}})^{\frac{N}{N-2}})\,dx \times b_0^{-\frac{t(N-2)}{N}}\|\Delta u\|_{N/2}^t \tag{2.3}$$

hence

$$\|u\|_{L^t(\Omega)} \leq \Gamma(t(N-2)/N+1)^{1/t}C^{1/t}b_0^{-(N-2)/N}|\Omega|^{1/t}\|\Delta u\|_{L^{N/2}(\Omega)}$$

We conclude using the Stirling's formula,

$$\Gamma(\frac{t(N-2)}{N}+1)^{1/t} \sim (\frac{N-2}{Ne})^{\frac{N-2}{N}}t^{\frac{N-2}{N}}.$$

□

Lemma 2.2.
$$\lim_{p\to\infty} c_p p^{\frac{N-2}{N}} = (\frac{Nb_0 e}{N-2})^{\frac{N-2}{N}} \tag{2.4}$$

Proof. Let L such that $B_L \subset \Omega$, and $l \in (0, L)$ to be fixed later. Let $m_l(x) = H((\log L/l)^{-1}\log 1/|x|)$, a regularized version of a Moser's function, where H is such that for $\epsilon \in (0, 1/2)$

$$H(t) = \begin{cases} \epsilon\Phi(t/\epsilon) & 0 < t \le \epsilon \\ t, & \epsilon < t \le 1-\epsilon \\ 1-\epsilon\Phi((1-t)/\epsilon) & 1-\epsilon < t \le 1, \\ 1 & 1 < t \end{cases}$$

with $\Phi \in C^\infty[0,1]$, $\Phi(0) = \Phi'(0) = 0$ $\Phi(1) = \Phi'(1) = 1$. Clearly $m_l \in W_0^{2,N/2}(\Omega)$ and $m_l(x) = 1$ for $|x| \in l$. A calculation gives

$$|\Delta m_l(x)| = \left| -\frac{a_0}{\omega_{N-1}} H'((\log L/l)^{-1}\log 1/|x|)(\log L/l)^{-1}|x|^{-2} + O(\log L/l)^{-2}|x|^{-2}) \right|$$

where $a_0 = (\omega_{N-1} b_0/N)^{\frac{N-2}{N}}$. Thus

$$\int_B |\Delta m_l|^{N/2} dx = M^{\frac{N}{2}} := \omega_{N-1}^{1-\frac{N}{2}} a_0^{\frac{N}{2}} (\log 1/r)^{1-\frac{N}{2}} A,$$

where $A \le 1 + C\epsilon$, see [1] for details. We define $\psi_l = m_l/M$ and find

$$\left(\int_\Omega \psi_l^{p+1}\, dx\right)^{\frac{1}{p+1}} \ge \left(\int_{B_l} \psi_l^{p+1}\, dx\right)^{\frac{1}{p+1}} = \frac{1}{M}(\frac{1}{N} l^N \omega_{N-1})^{\frac{1}{p+1}}.$$

Take $l = L\exp(-(N-2)(p+1)/N^2)$ and recall that $M^{-1} = (\omega_{N-1}\log L/l)^{\frac{N-2}{N}} A^{-\frac{2}{N}} a_0^{-1}$, we find

$$\int_\Omega \psi_l^{p+1} dx)^{\frac{1}{p+1}} \ge \frac{\omega_{N-1}^{\frac{N-2}{N}}}{a_0} (\frac{N-2}{N^2})^{\frac{N-2}{N}} (p+1)^{\frac{N-2}{N}} A^{-\frac{2}{N}} (\frac{1}{N} L^N \omega_{N-1})^{\frac{1}{p+1}} e^{-\frac{N-2}{N}}.$$

Then

$$c_p (p+1)^{\frac{N-2}{N}} \le a_0 \omega_{N-1}^{-\frac{N-2}{N}} (\frac{N-2}{N^2})^{-\frac{N-2}{N}} A^{\frac{2}{N}} (\frac{1}{N} L^N \omega_{N-1})^{-\frac{1}{p+1}} e^{\frac{N-2}{N}} \tag{2.5}$$

$$\le b_0^{\frac{N-2}{N}} (\frac{Ne}{N-2})^{\frac{N-2}{N}} A^{\frac{2}{N}} (\frac{1}{N} L^N \omega_{N-1})^{-\frac{1}{p+1}} \tag{2.6}$$

Letting $p \to \infty$ and $\epsilon \to 0$, we obtain the result by combining this with Lemma 2.1. $\square$

Corollary 2.3. *We have*

$$p^{\frac{N-2}{2}} \int_\Omega |\Delta u_p|^{N/2} dx = (\frac{Nb_0 e}{N-2})^{\frac{N-2}{2}}, \quad p^{\frac{N-2}{2}} \int_\Omega u_p^{p+1} dx = (\frac{Nb_0 e}{N-2})^{\frac{N-2}{2}}.$$

Proof of Theorem 1.1. Let Λ_N be given by the minimization problem

$$\Lambda_N = \inf \left\{ \frac{\|\Delta u\|_{L^{\frac{N}{2}}(\Omega)}}{\|u\|_{L^{\frac{N}{2}}(\Omega)}} \mid u \in W^{2,\frac{N}{2}}(\Omega) \cap W_0^{1,\frac{N}{2}}(\Omega) \right\}.$$

From Lemma 2.1, we find $0 < \Lambda_N < \infty$. Using this,

$$\int_\Omega u_p^{p+1}\, dx = \int_\Omega |\Delta u_p|^{\frac{N}{2}}\, dx \geq \Lambda_N^{N/2} \int_\Omega u_p^{N/2}\, dx.$$

Thus $\int_\Omega (u_p^{p+1} - \Lambda_N^{N/2} u_p^{N/2}) \geq 0$, therefore $\|u_p\|_{L^\infty(\Omega)}^{p+1-\frac{N}{2}} \geq \Lambda_N^{\frac{N}{2}}$. If $p > (N-2)/2$ then

$$\|u_p\|_{L^\infty(\Omega)} > \Lambda_N^{\overline{\frac{N}{2(p+1-\frac{N}{2})}}} \geq C_1 > 0.$$

To obtain an upper bound for $\|u_p\|_{L^\infty(\Omega)}$, let

$$\gamma_p = \max_{x\in\Omega}, \quad \Omega_l := \{x \in \Omega \colon t < u_p(x)\}, \quad \mathcal{A} = \{x \in \Omega \colon \frac{\gamma_p}{2} < u_p(x)\}. \tag{2.7}$$

Applying Lemma 2.1 and 2.3, we obtain

$$(\int_\Omega u_p^{\frac{N^2 p}{2(N-2)}} dx)^{\frac{2(N-2)}{N^2 p}} \leq D_{\frac{N^2 p}{2(N-2)}} (\frac{N^2 p}{2(N-2)})^{\frac{N-2}{N}} \|\Delta u_p\|_{L^{\frac{N}{2}}(\Omega)} \leq M$$

where M is independent of p for p large. This implies

$$(\frac{\gamma_p}{2})^{\frac{N^2 p}{2(N-2)}} |\mathcal{A}| \leq M^{\frac{N^2 p}{2(N-2)}}. \tag{2.8}$$

Taking $v_p^{\frac{2}{N-2}} = -\Delta u_p$ and integrating by parts

$$\int_{\partial\Omega_t} |\nabla u_p|\, ds = \int_{\Omega_t} v_p^{\frac{2}{N-2}} dx.$$

By Coarea formula we have

$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{1}{|\nabla u_p|}\, ds$$

Then Schwartz inequality gives

$$-\frac{d}{dt}|\Omega_t| \int_{\Omega_t} v_p^{\frac{2}{N-2}}\, dx = \int_{\partial\Omega_t} |\nabla u_p|\, ds \int_{\partial\Omega_t} \frac{1}{|\nabla u_p|}\, ds \geq |\partial\Omega_t|^2.$$

The isoperimetric inequality in $\mathbb{R}^N$

$$|\partial\Omega_t| \geq C_N |\Omega_t|^{\frac{N-1}{N}},$$

yields

$$-\frac{d}{dt}|\Omega_t| \int_{\Omega_t} v_p^{\frac{2}{N-2}}\, dx \geq C_N^2 |\Omega_t|^{\frac{2(N-1)}{N}}.$$

We define $r(t)$ such that $|\Omega_t| = \omega_{N-1} r^N(t)/N$. Then

$$\frac{d}{dt}|\Omega_t| = \omega_{N-1} r^{N-1}(t) r'(t) < 0.$$

Thus

$$\frac{N}{\omega_{N-1}^{\frac{N-2}{N}} C_N^2 r(t)^{N-1}} \int_{\Omega_t} v_p^{\frac{2}{N-2}}\, dx \geq -\frac{1}{r'(t)}$$

$$-\frac{dt}{dr} \leq \frac{N}{\omega_{N-1}^{\frac{N-2}{N}} C_N^2 r(t)^{N-1}} \int_{\Omega_t} v_p^{\frac{2}{N-2}}\, dx \leq \frac{N}{\omega_{N-1}^{\frac{N-2}{N}} C_N^2 r(t)^{N-1}} (\sup_\Omega v_p)^{\frac{2}{N-2}} |\Omega_t|$$

$$= (\sup_\Omega v_p)^{\frac{2}{N-2}} \frac{r(t)\omega_{N-1}^{\frac{2}{N}}}{N C_N^2}.$$

Integrating this inequality from $r = 0$ to $r = r_0$, we have

$$t(0) - t(r_0) \leq \frac{\omega_{N-1}^{\frac{2}{N}}}{2N C_N^2} r_0^2 (\sup_\Omega v_p)^{\frac{2}{N-2}}$$

Choosing r_0 such that $t(r_0) = \gamma_p/2$ that is $|\mathcal{A}| = |\Omega_{\gamma_p/2}| = \omega_{N-1} r_0^N$, the last inequality yields

$$\gamma_p \leq \frac{\omega_{N-1}^{\frac{2}{N}}}{N C_N^2} r_0^2 (\sup_\Omega v_p)^{\frac{2}{N-2}}$$

But $v_p^{\frac{2}{N-2}} = -\Delta u_p$ satisfies

$$-\Delta v_p = u_p^p \quad \text{in} \quad \Omega, \quad v_p = 0 \quad \text{on} \quad \partial\Omega.$$

By elliptic regularity ([9], Theorem 3.7), $\sup_\Omega v_p \leq C \sup_\Omega u_p^p \leq C \gamma_p^p$ where $C = C(\Omega)$. Thus we have

$$\gamma_p \leq \bar{C} \omega_{N-1}^{\frac{2}{N}} \gamma_p^{\frac{2p}{N-2}} r_0^2 = \bar{C} \gamma_p^{\frac{2p}{N-2}} |\mathcal{A}|^{\frac{2}{N}} \tag{2.9}$$

where $\bar{C} = C^{\frac{2p}{N-2}}/(N C_N^2)$. By (2.8) and (2.9),

$$\gamma_p \leq \bar{C} \gamma_p^{-p} (2M)^{\frac{Np}{N-2}}$$

which implies

$$\gamma_p \leq \bar{C}^{\frac{1}{p+1}} (2M)^{\frac{Np}{(N-2)(p+1)}}.$$

Therefore there exists $C > 0$ independent of p such that $\gamma_p \leq C$ for p large. □

Next we have the corollary.

Corollary 2.4. *There exist $C_1, C_2 > 0$ independent of p such that*

$$\frac{C_1}{p^{\frac{N-2}{2}}} \leq \int_\Omega u_p^p\, dx \leq \frac{C_2}{p^{\frac{N-2}{2}}}$$

for large p.

Proof. From Corollary 2.3 and Theorem 1.1, we have for p large

$$C' \leq p^{\frac{N-2}{2}} \int_\Omega u_p^{p+1}\, dx \leq \|u_p\|_{L^\infty(\Omega)} p^{\frac{N-2}{2}} \int_\Omega u_p^p\, dx \leq C'' p^{\frac{N-2}{2}} \int_\Omega u_p^p\, dx,$$

where $C', C'' > 0$ constant independent of p. This shows the left inequality. Now by Holder inequality,

$$p^{\frac{N-2}{2}} \int_\Omega u_p^p\, dx \leq (p^{\frac{N-2}{2}} \int_\Omega u_p^{p+1}\, dx)^{\frac{p}{p+1}} p^{\frac{N-2}{2(p+1)}} |\Omega|^{\frac{1}{p+1}}.$$

Using Corollary 2.3, for p large the RHS is bounded. □

3. Proof of Theorem 1.2

Let

$$w_p := \frac{u_p}{(\int_\Omega u_p^p\, dx)^{\frac{2}{N-2}}} = \frac{u_p}{\lambda_p}, \quad \lambda_p := (\int_\Omega u_p^p\, dx)^{\frac{2}{N-2}},$$

and

$$f_p(x) := \frac{u_p^p}{\int_\Omega u_p^p\, dx}.$$

This yields

$$\begin{aligned} -\Delta(-\Delta w_p)^{(N-2)/2} &= f_p \quad w_p > 0 \quad \text{in} \quad \Omega, \\ w_p &= \Delta w_p = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

Since Ω is convex, we can derive standard uniform boundary estimates of $\{w_p\}$ which leads to conclude that the blow up set of $\{w_p\}$ is contained in the interior of Ω.

Using the methods in [11] Proposition 3.2, we can show that

$$a) \int_\Omega w_p \phi_1 \leq C \quad \text{and} \quad b) \int_\Omega (-\Delta w_p)\phi_1 \leq C,$$

where ϕ_1 is the positive eigenvalue of $(-\Delta, H_0^1(\Omega))$, normalized to $\int_\Omega \phi_1 = 1$. Combining inequality $a)$ with the ideas of [6] based on the method of the moving planes from [8], we obtain a uniform bound in the boundary. Indeed, we can find $\delta > 0$ such that for any $x \in \Omega_\delta := \{z \in \bar{\Omega} \colon d(z, \partial\Omega) < \delta\}$, we have

$$w_p(x) \leq C(\Omega_\delta).$$

Now we extend a known results from [4].

Lemma 3.1. *Let u be a regular solution of*

$$-\Delta(-\Delta u)^{\frac{N-2}{2}} = f(x) \quad in \quad \Omega \subset \mathbb{R}^N \tag{3.1}$$

$$u = \Delta u = 0 \quad on \quad \partial\Omega, \tag{3.2}$$

where $f \in L^1(\Omega)$, $f \geq 0$. For any $\epsilon \in (0, b_0)$ we have

$$\int_\Omega \exp\left(\frac{(b_0-\delta)|u(x)|}{\|f\|_{L^1(\Omega)}^{\frac{2}{N-2}}}\right)\, dx \leq \frac{b_0}{\delta}|\Omega|.$$

Proof. We proof this by the symmetrization method. Consider the symmetrized problem

$$-\Delta(-\Delta U)^{\frac{N-2}{2}} = F(x) \quad \text{in} \quad \Omega^* \tag{3.3}$$
$$U = \Delta U = 0 \quad \text{on} \quad \partial\Omega^*. \tag{3.4}$$

Here Ω^* is a ball centered at the origin with the same volume at Ω, say $\Omega^* = B(0,R)$, and F is the symmetric decreasing rearrangement of f. By [16] and [17], we have

$$u^* \leq U$$

where u^* is the symmetric rearrangement of u. Clearly U satisfies

$$-(r^{N-1}U'(r))' = r^{N-1}V^{\frac{2}{N-2}}, \quad r \in (0,R) \tag{3.5}$$
$$-(r^{N-1}V'(r))' = r^{N-1}F(r), \quad r \in (0,R) \tag{3.6}$$
$$U'(0) = V'(0) = U(R) = V(R) = 0. \tag{3.7}$$

Multiple integrations give,

$$-U'(r) \leq \frac{1}{(N-2)^{\frac{N}{N-2}}\omega_{N-1}^{\frac{2}{N-2}}}\frac{1}{r}\|F\|_{L^1(\Omega^*)}^{\frac{2}{N-2}}.$$

Hence

$$|U(r)| \leq \frac{1}{(N-2)^{\frac{N}{N-2}}\omega_{N-1}^{\frac{2}{N-2}}}\|F\|_{L^1(\Omega^*)}^{\frac{2}{N-2}} \log(\frac{R}{r}), \tag{3.8}$$

$$\int_{\Omega^*} \exp\left[(N-\epsilon)(N-2)^{\frac{N}{N-2}}\omega_{N-1}^{\frac{2}{N-2}}\frac{U}{\|F\|_{L^1(\Omega^*)}^{\frac{2}{N-2}}}\right] \leq \int_{B_R(0)} \exp\log(\frac{R}{r})^{N-\epsilon}dr \tag{3.9}$$

$$= \omega_{N-1}\int_0^R (\frac{R}{r})^{N-\epsilon}r^{N-1}\, dr = \epsilon^{-1}\omega_{N-1}R^N. \tag{3.10}$$

Letting $\epsilon(N-2)^{\frac{N}{N-2}}\omega_{N-1}^{\frac{2}{N-2}} = \delta$, we have

$$\int_{\Omega^*} \exp\left[(b_0-\delta)\frac{U}{\|F\|_{L^1(\Omega^*)}^{\frac{2}{N-2}}}\right] \leq \frac{b_0}{\delta}|\Omega|.$$

By the properties of the symmetric decreasing functions, $\|F\|_{L^1(\Omega^*)} = \|f\|_{L^1(\Omega)}$, and

$$\int\limits_{\Omega} \exp\left[(b_0-\delta)\frac{u(x)}{\|f\|_{L^1(\Omega)}^{\frac{2}{N-2}}}\right] dx = \int\limits_{\Omega} \exp\left[(b_0-\delta)\frac{u^*(x)}{\|f\|_{L^1(\Omega)}^{\frac{2}{N-2}}}\right] dx$$

$$\leq \int\limits_{\Omega^*} \exp\left[(b_0-\delta)\frac{U}{\|F\|_{L^1(\Omega^*)}^{\frac{2}{N-2}}}\right] \leq \frac{b_0}{\delta}|\Omega|,$$

which proves the lemma. □

Corollary 3.2. *a) Let u_n be a sequence of solutions of*

$$-\Delta(-\Delta u_n)^{\frac{N-2}{2}} = V_n e_n^u \quad \textit{in} \quad \Omega \subset \mathbb{R}^N$$
$$u_n = \Delta u_n = 0 \quad \textit{on} \quad \partial\Omega,$$

such that $\|V_n\|_{L^p(\Omega)} \leq C_1$, for some $p \in (1,\infty)$, $\|V_n\|_{L^p(\Omega)} \leq C_1$, and

$$\|V_n e^{u_n}\|_{L^1(\Omega)} \leq \epsilon_0 < b_o p/(p-1).$$

Then $\{u_n\}$ is uniformly bounded in $L^\infty_{loc}(\Omega)$.

b) Let u_n be a sequence of solutions of $-\Delta(-\Delta u_n)^{\frac{N-2}{2}} = V_n e^{u_n}$ in $\Omega \subset \mathbb{R}^N$ with $V_n \geq 0$ and $u_n, -\Delta u_n \geq 0$ on the boundary. Assume for some $p \in (1,\infty)$ that

$$(1) \quad \|V_n\|_{L^p(\Omega)} \leq C_1, \tag{3.11}$$
$$(2) \quad \|u_n\|_{L^1(\Omega)} \leq C_2 \tag{3.12}$$
$$(3) \quad \|V_n e^{u_n}\|_{L^1(\Omega)} \leq \epsilon_0 < b_o p/(p-1) \tag{3.13}$$

Then $\{u_n\}$ is uniformly bounded in $L^\infty_{loc}(\Omega)$.

Proof. Part a). Fix $\delta > 0$, so that $b_0 - \delta > \epsilon_0(p' + \delta)$. By Lemma 3.1, we have

$$\int\limits_{\Omega} \exp[(p'+\delta)|u_n|] \leq C$$

for some C independent of n. Therefore e^{u_n} is bounded in $L^{p'+\delta}(\Omega)$, hence $V_n e^{u_n}$ is bounded in $L^{1+\epsilon_0}(\Omega)$. Then by elliptic regularity, we have u_n bounded in $L^\infty_{loc}(\Omega)$.

The part b) follows similarly. With restriction we may assume that $\Omega = B_R(x_0)$ for some x_0. We consider

$$-\Delta(-\Delta u_{1n})^{\frac{N-2}{2}} = V_n e^{u_n} \quad \text{in} \quad \Omega \subset \mathbb{R}^N$$
$$u_{1n} = \Delta u_{1n} = 0 \quad \text{on} \quad \partial\Omega,$$

and $-\Delta(-\Delta u_{2n})^{\frac{N-2}{2}} = 0$ in Ω with $u_{2n} = u_n \geq 0$ and $-\Delta u_{2n} = -\Delta u_n \geq 0$. By uniqueness

$$(-\Delta u_n)^{\frac{N-2}{2}} = (-\Delta u_{1n})^{\frac{N-2}{2}} + (-\Delta u_{2n})^{\frac{N-2}{2}}.$$

where each term is positive. For $N=3$, we have $-\Delta u_n \le -2(\Delta u_{1n}+\Delta u_{2n})$ and for $N\ge 4$, $-\Delta u_n \le -(\Delta u_{1n}+\Delta u_{2n})$. In the last case, define $H=u_n-u_{1n}-u_{2n}$, and by the maximum principle $H\le 0$ in $B_R(x_0)$. This gives

$$u_n \le u_{1n}+u_{2n} \quad \text{in} \quad B_R(x_0),$$

and similarly for $N=3$, we have

$$u_n \le 2(u_{1n}+u_{2n}) \quad \text{in} \quad B_R(x_0).$$

Note that $u_{2n}\le u_n$ and $u_{1n}\le u_n$ in $B_R(x_0)$. Now a uniform bound for u_{1n} is given by part a) and we know by mean value theorem,

$$\|u_{2n}\|_{L^\infty(B_{R/2}(x_0))} \le C\|u_{2n}\|_{L^1(B_R(x_0))} \le C\|u_n\|_{L^1(\Omega)} \le C,$$

and the last inequality follows from the assumption (2). □

Let u_n, w_n, λ_n, and f_n denote u_{p_n}, w_{p_n}, λ_{p_n}, and f_{p_n}. First we note that the blow up set S of the sequence $\{w_n\}$ is not empty. In fact,

$$\sup_{x\in\Omega} w_n(x) \ge \frac{C}{\lambda_n} \to +\infty,$$

by Theorem 1.1 and using that $p_n\lambda_n \le C$ for C independent of p_n large. This also shows that the set peaks of $\{u_n\}$ is contained in the set S. Since

$$f_n(x) = \frac{u_n^{p_n}}{\int\limits_\Omega u_n^{p_n}\,dx} \in L^1(\Omega), \quad f_n \ge 0, \quad \int\limits_\Omega f_n(x)\,dx = 1,$$

there exists a subsequence (denoted also by $\{u_n\}$) such that there exists a positive bounded measure μ in the set of real bounded Borel measures in Ω, satisfying $\mu(\Omega)\le 1$ and

$$\int\limits_\Omega f_n\phi \to \int\limits_\Omega \phi\,d\mu \quad \text{for all} \quad \phi\in C_0(\Omega).$$

We now define the quantity

$$L_0 = \frac{1}{e}\limsup_{p\to\infty} p\left(\int\limits_\Omega u_p^p\,dx\right)^{\frac{2}{N-2}}.$$

From the proof of Theorem 1.1, we obtain $1\le L_0 \le Nb_0/(N-2)$.

For any $\delta>0$ we call a point $x_0\in\Omega$ a $\delta-$regular point of $\{u_n\}$ if there exists $\varphi\in C_0(\Omega)$, $0\le\varphi\le 1$ with $\varphi\equiv 1$ in a neighborhood of x_0 such that

$$\int\limits_\Omega \varphi d\mu \le \left(\frac{b_0}{L_0+2\delta}\right)^{\frac{N-2}{2}}.$$

We also define for $\delta>0$, $\delta-$irregular set of a sequence $\{u_n\}$ such that

$$\Sigma(\delta) = \{x_0\in\Omega\colon x_0 \text{ is not a } \delta-\text{regular point }\}.$$

Note that $x_0 \in \Sigma(\delta)$ implies

$$\mu(x_0) > \left(\frac{b_0}{L_0 + 2\delta}\right)^{\frac{N-2}{2}}.$$

The next results is crucial to prove Theorem 1.2.

Lemma 3.3. *Assume $N > 3$. Let x_0 be a $\delta-$regular point of a sequence $\{u_n\}$ then $\{w_n\}$ is bounded in $L^\infty(B_{R_0}(x_0))$ for some $R_0 > 0$.*

Proof. Let x_0 be a $\delta-$regular point. Then there exists $R > 0$ such that

$$\int_{B_R(x_0)} f_n \, dx < (\frac{b_0}{L_0 + \delta})^{\frac{N-2}{2}}$$

for n sufficiently large.

Let w_{1n} be solution of

$$\begin{aligned} -\Delta(-\Delta w_{1n})^{\frac{N-2}{2}} &= f_n \quad \text{in} \quad B_R(x_0) \\ w_{1n} = \Delta w_{1n} &= 0 \qquad \text{on} \quad \partial B_R(x_0), \end{aligned}$$

and let w_{2n} be solution of

$$\begin{aligned} -\Delta(-\Delta w_{2n})^{\frac{N-2}{2}} &= 0 \quad \text{in} \quad B_R(x_0) \\ w_{2n} = w_n, \quad \Delta w_{2n} &= \Delta w_n \qquad \text{on} \quad \partial B_R(x_0), \end{aligned}$$

By the maximum principle we have $-\Delta w_{1n} > 0$ and $-\Delta w_{2n} > 0$, $w_{1n} > 0$, and $w_{2n} > 0$ in $B_R(x_0)$.

Clearly by uniqueness

$$(-\Delta w_n)^{\frac{N-2}{2}} = (-\Delta w_{1n})^{\frac{N-2}{2}} + (-\Delta w_{2n})^{\frac{N-2}{2}}.$$

If $N \geq 4$, $-\Delta w_n \leq -(\Delta w_{1n} + \Delta w_{2n})$, then we can define $H = w_n - w_{1n} - w_{2n}$, and by the maximum principle $H \leq 0$ in $B_R(x_0)$. This gives

$$w_n \leq w_{1n} + w_{2n} \quad \text{in} \quad B_R(x_0)$$

Note that $w_{2n} \leq w_n$ and $w_{1n} \leq w_n$ in $B_R(x_0)$. The solution w_{2n} is uniformly bounded near x_0, in fact the the mean value theorem gives

$$\|w_{2n}\|_{L^\infty(B_{R/2}(x_0))} \leq C\|w_{2n}\|_{L^1(B_R(x_0))} \leq C\|w_n\|_{L^1(\Omega)} \leq C,$$

and the last inequality follows from Lemma 3.1.

So we need to bound $\{w_{1n}\}$. We first choose t such that $t' := t/(t-1) = L_0 + \delta/2$. Since $L_0 > 1$, there exists $t > 1$. Then we have

$$\int_{B_R(x_0)} f_n \, dx < (\frac{b_0}{L_0 + \delta})^{\frac{N-2}{2}}$$

Lemma 3.1 implies

$$\int_{B_R(x_0)} \exp(t'|w_{1n}(x)|) \, dx = \int_{B_R(x_0)} \exp((L_0 + \delta/2)|w_{1n}(x)|) \, dx \leq C, \tag{3.14}$$

where $C = C(\delta) \to \infty$ as $\delta \to 0$.

By the inequality $\log x \le x/e$ for $x > 0$, we have

$$\log f_n = \log \frac{u_n^{p_n}}{\lambda_n^{\frac{N-2}{2}}} = p_n \log \frac{u_n}{\lambda_n^{\frac{N-2}{2p_n}}} \le p_n \frac{u_n}{e\lambda_n^{\frac{N-2}{2p_n}}} \tag{3.15}$$

$$\le \frac{L_0 + \delta/3}{\lambda_n} \frac{u_n}{\lambda_n^{\frac{N-2}{2p_n}}} = \frac{t' - \delta/6}{\lambda_n^{\frac{N-2}{2p_n}}} \frac{u_n}{\lambda_n} \le t' w_n(x). \tag{3.16}$$

The second inequality follows from the definition of L_0 and the last form $\lim_{n\to\infty} \lambda_n^{\frac{N-2}{2p_n}} = 1$.

Thus we get the pointwise estimate $f_n(x) < \exp(t' w_n(x))$, which implies

$$(f_n \exp(-w_{1n}(x))^t < C \exp(t' w_{1n}(x)) \tag{3.17}$$

in $B_{R/2}(x_0)$ because w_{2n} is uniformly bounded in $B_{R/2}(x_0)$ and $w_n \le w_{1n} + w_{2n}$ in $B_{R/2}(x_0)$.

Rewrite the equation for w_{1n} as

$$-\Delta(-\Delta w_{1n})^{\frac{N-2}{2}} = \underline{f_n e^{-w_{1n}(x)}} e^{+w_{1n}(x)} \quad \text{in} \quad B_R(x_0)$$
$$w_{1n} = \Delta w_{1n} = 0 \quad \text{on} \quad \partial B_R(x_0),$$

Clearly $w_{1n}, -\Delta w_{1n} \ge 0$ in $B_R(x_0)$ We now check the assumptions of Corollary 3.2. Let $V_n = f_n e^{-w_n(x)}$,

$$(1) \quad \|V_n\|_{L^t(B_{R/2}(x_0))} \le C_1, \quad \text{by (3.14) and (3.17)} \tag{3.18}$$
$$(2) \quad \|w_{1n}\|_{L^1(B_{R/2}(x_0))} \le C_2 \quad \text{by Lemma 3.1} \tag{3.19}$$
$$(3) \quad \|V_n e^{u_n}\|_{L^1(B_{R/2}(x_0))} = \|f_n\|_{L^1(B_{R/2}(x_0))} \le \epsilon_0 < Ct/(t-1) \tag{3.20}$$

Applying Corollary 3.2, we conclude that $\{w_{1n}\}$ is uniformly bounded in $B_{R/2}(x_0)$. □

Lemma 3.4. *$S = \Sigma(\delta)$ for any $\delta > 0$.*

Proof. $S \subset \Sigma(\delta)$ is clear from Lemma 3.3. Now suppose that $x_0 \in \Sigma(\delta)$ and $\|w_n\|_{L^\infty(B_{R_0}(x_0))} < C$ for some C independent of n. then $f_n = \lambda_n^{p_n-1} w_n^{p_n} \to 0$ uniformly on $B_{R_0}(x_0)$, which implies x_0 is a $\delta-$regular point, that is $x_0 \notin \Sigma(\delta)$. Thus contradiction shows that for every $R > 0$ we have $\lim_{n\to\infty} \|w_n\|_{L^\infty(B_R(x_0))} = \infty$ at least for a subsequence. So $x_0 \in S$. □

This lemma implies that

$$1 \ge \mu(\Omega) \ge \left(\frac{b_0}{L_0 + 2\delta}\right)^{\frac{N-2}{2}} \text{card}(\Sigma(\delta)) = \left(\frac{b_0}{L_0 + 2\delta}\right)^{\frac{N-2}{2}} \text{card}(S).$$

Combining this with the estimate $L_0 \le Nb_0/(N-2)$, we have

$$\text{card}(S) \le \left(\frac{L_0 + 2\delta}{b_0}\right)^{\frac{N-2}{2}} \le \left(\frac{Nb_0/(N-2) + 2\delta}{b_0}\right)^{\frac{N-2}{2}}$$

hence, since S is not empty

$$1 \leq \operatorname{card}(S) \leq \left(\frac{N}{N-2}\right)^{\frac{N-2}{2}} < e.$$

This proves the first part of Theorem 1.2.

In the following we assume $\operatorname{card}(S) = 1$ with $S = \{x_0\}$. Then $w_n(x) \leq C$ on any compact set $K \subset \overline{\Omega} \setminus \{x_0\}$, which implies $f_n \to 0$ uniformly on compacts of $\overline{\Omega} \setminus \{x_0\}$.

Take $\varphi \in C_0(\Omega)$. For given $\epsilon > 0$ we choose $r > 0$ small such that as $n \to \infty$, we have

$$|\int_\Omega f_n \varphi\, dx - \varphi(x_0)| \leq \int_\Omega f_n |\varphi(x) - \varphi(x_0)|\, dx$$

$$\leq \int_{B_r(x_0)} f_n |\varphi(x) - \varphi(x_0)|\, dx + \int_{\Omega \setminus B_r(x_0)} f_n |\varphi(x) - \varphi(x_0)|\, dx \leq \epsilon.$$

Therefore

$$f_n \to \delta_{x_0} \tag{3.21}$$

in the sense of distributions. Let $\tilde{w}_n^{\frac{2}{N-2}} = -\Delta w_n$, then

$$-\Delta \tilde{w}_n = f_n \quad \text{in} \quad \Omega, \quad \tilde{w}_n = 0 \quad \text{on} \quad \partial\Omega.$$

with $f_n \to 0$ uniformly in compact subsets of $\overline{\Omega} \setminus \{x_0\}$. This proves 1) of Theorem 1.2.

For 2), on any compact $K \subset \overline{\Omega} \setminus \{x_0\}$, we have $\tilde{w}_n$ is bounded and $f_n \to 0$ uniformly. By elliptic regularity there exists a subsequence of $\tilde{w}_n$, still denoted by $\tilde{w}_n$ that approaches a function say G' in $C^{2,\alpha}(K)$ weakly in $W^{1,q}(\Omega)$ for $(1 < q < 2)$ and strongly in $L^1(\Omega)$ by the compact embedding $W^{1,q}(\Omega) \hookrightarrow L^1(\Omega)$. As in [12], $G' = G(\cdot, x_0)$. Since $-\Delta w_n = \tilde{w}_n^{\frac{2}{N-2}}$ by the convergence of $\tilde{w}_n$, we have that w_n converges to G'' in $C^{2,\alpha}(\Omega)$, and by uniqueness $G'' = \tilde{G}$.

To prove 3) we use a Pohozaev identity. From [10, 18], for any $y \in \mathbb{R}^N$, we have for any $\Omega' \subset \mathbb{R}^n$, the following identity

$$\int_{\Omega'} \Delta u(x-y, \nabla v) + \Delta v(x-y, \nabla u) - (N-2)(\nabla u, \nabla v) dx =$$

$$\int_{\partial\Omega'} \frac{\partial u}{\partial n}(x-y, \nabla v) + \frac{\partial v}{\partial n}(x-y, \nabla u) - (\nabla u, \nabla v)(x-y, n)\, ds. \tag{3.22}$$

Let $\Omega' = \Omega$. For the system $-\Delta v = u^p$ and $-\Delta u = v^{\frac{2}{N-2}}$ in Ω, the identity (3.22) takes the form

$$\left(\frac{N}{p+1} - \bar{a}\right) \int_\Omega u^{p+1}\, dx + (N-2-\bar{b}) \int_\Omega v^{\frac{N}{N-2}}\, dx \tag{3.23}$$

$$+(N-2-\bar{a}-\bar{b}) \int_\Omega (\nabla u, \nabla v)\, dx = -\int_{\partial\Omega} (\nabla u, \nabla v)(x-y, n)\, ds. \tag{3.24}$$

We choose $\bar{a}+\bar{b}=N-2$, $\bar{a}=0$ and so $\bar{b}=N-2$. This gives for u_p and $v_p=(-\Delta u_p)^{\frac{N-2}{2}}$,

$$\frac{N}{p+1}\int_{\Omega} u_p^{p+1}\,dx = -\int_{\partial\Omega}\frac{\partial u_p}{\partial n}\frac{\partial v_p}{\partial n}(n,x-y)\,ds \tag{3.25}$$

differentiating with respect to y,

$$\int_{\partial\Omega}\frac{\partial u_p}{\partial n}\frac{\partial v_p}{\partial n}n\,ds=0.$$

Taking $v_p\to G$ and $u_p\to\tilde{G}$ in $C^2(\Omega)$, as $p\to\infty$, we get

$$\int_{\partial\Omega}(\nabla\tilde{G}(x,x_0),\nabla G(x,x_0))n\,ds=0 \tag{3.26}$$

On the other hand we have the following result.

Lemma 3.5. *For every $x_0\in\Omega$*

$$\int_{\partial\Omega}(\nabla\tilde{G}(x,x_0),n)(\nabla(\Delta\tilde{G}(x,x_0))^{\frac{N-2}{2}},n)n\,ds=-\nabla\tilde{\phi}(x_0). \tag{3.27}$$

Hence combining (3.26) with (3.27), we complete the proof of part 3) and the Theorem 1.2 is proven.

Proof Lemma 3.5. Let $\Omega'=\Omega\setminus B_r$ with $r>0$. For a system $-\Delta v=0$ and $-\Delta u=v^{\frac{2}{N-2}}$ in Ω', the identity 3.22, takes the form

$$\int_{\Omega'}(N-2)v^{\frac{N}{N-2}}-\bar{a}v^{\frac{N}{N-2}}\,dx=\int_{\partial\Omega'}\frac{N-2}{N}v^{\frac{N}{N-2}}(x-y,n)\,ds$$
$$+\int_{\partial\Omega'}\frac{\partial u}{\partial n}\left[(x-y,\nabla v)+\bar{a}v\right]+\frac{\partial v}{\partial n}\left[(x-y,\nabla u)+\bar{b}u\right]-(\nabla u,\nabla v)(x-y,n)\,ds \tag{3.28}$$

with $\bar{a}+\bar{b}=N-2$. We choose $\bar{a}=N-2$ and take $v=G(x,0)$ and $u=\tilde{G}(x,0)$. Upon differentiation with respect to y, (3.28) transforms into

$$\int_{\partial\Omega}\frac{\partial\tilde{G}}{\partial n}\frac{\partial G}{\partial n}n\,ds=\int_{\partial B_r}\left\{\frac{N-2}{N}G^{\frac{N}{N-2}}n+\frac{\partial\tilde{G}}{\partial n}\nabla G+\frac{\partial G}{\partial n}\nabla\tilde{G}-(\nabla\tilde{G},\nabla G)n\right\}ds.$$

Note that $u=v=0$ on $\partial\Omega$, implies $\nabla u=(\nabla u,n)n$ and $\nabla v=(\nabla v,n)n$ on $\partial\Omega$. Let $\Gamma=\omega_{N-1}(N-2)$, we have

$$\nabla\tilde{G}=-\frac{1}{\Gamma^{\frac{2}{N-2}}(N-2)}|x|^{-2}x+\nabla\tilde{g},\quad \nabla G=-\frac{1}{\omega_{N-1}}|x|^{-N}x+\nabla g,$$

$$\frac{\partial\tilde{G}}{\partial n}=-\frac{1}{\Gamma^{\frac{2}{N-2}}(N-2)}|x|^{-1}+(\nabla\tilde{g},n),\quad \frac{\partial G}{\partial n}=-\frac{1}{\omega_{N-1}}|x|^{1-N}+(\nabla g,n)$$

$$(\nabla\tilde{G},\nabla G)=\frac{|x|^{-N}}{\omega_{N-1}\Gamma^{\frac{2}{N-2}}(N-2)}-\frac{(\nabla g,x)}{\Gamma^p(N-2)}|x|^{-2}-\frac{(\nabla\tilde{g},x)}{\omega_{N-1}}|x|^{-N}+(\nabla\tilde{g},\nabla g)$$

and

$$\frac{N-2}{N}G^{\frac{N}{N-2}}=\frac{N-2}{N}\left[\frac{1}{\Gamma^{\frac{2}{N-2}}}|x|^{-2}-\Delta\tilde{g}\right]\left[\frac{1}{\Gamma}|x|^{2-N}+g\right]$$

Using $\int_{\partial B_r} n=0$, we get

$$\begin{aligned}\int_{\partial\Omega}\frac{\partial\tilde{G}}{\partial n}\frac{\partial G}{\partial n}n\,ds=\frac{N-2}{Nr^{N-1}}\int_{\partial B_r}\left\{\frac{1}{\Gamma^p}r^{N-2-1}g-\Delta\tilde{g}\frac{1}{\Gamma}r-\Delta\tilde{g}gr^{N-1}\right\}n\,ds\\ +\frac{1}{r^{N-1}}\int_{\partial B_r}\{(\nabla\tilde{g},n)\nabla g+(\nabla g,n)\nabla\tilde{g}-(\nabla\tilde{g},\nabla g)n\}r^{N-1}\,ds\\ -\frac{1}{r^{N-1}}\int_{\partial B_r}\left\{\frac{1}{\omega_{N-1}}\nabla\tilde{g}+\frac{r^{N-2}}{\Gamma^p(N-2)}\nabla g\right\}ds.\end{aligned}\tag{3.29}$$

Since $N\geq 3$, and $\tilde{g}$ and g are regular, we obtain in the limit as $r\to 0$,

$$\int_{\partial\Omega}\frac{\partial\tilde{G}}{\partial n}\frac{\partial G}{\partial n}n\,ds=\lim_{r\to 0}\frac{1}{r^{N-1}}\int_{\partial B_r}\frac{1}{\omega_{N-1}}\nabla\hat{g}\,ds=\nabla\tilde{\phi}(0).$$

□

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